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ON THE SET COVERING POLYTOPE: II. LIFTING THE FACETS WITH COEFFICIENTS IN {0,1,2}

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Abstract

In an earlier paper [1] we characterized the class of facets of the set covering polytope defined by inequalities with coefficients equal to 0, 1 or 2. ~~In~~ this paper we connect that characterization to the theory of facet lifting. In particular, ~~we introduce~~ a family of lower dimensional polytopes and associated inequalities ced having only three nonzero coefficients, whose lifting yields all the valid inequalities in the above class, with the lifting coefficients given by closed form expressions.

1. Introduction

A great variety of practical optimization problems can be formulated as set covering problems, i.e. integer programs of the form

$$\text{(SC)} \qquad \min \{cx \mid Ax \geq 1,\ x \in \{0,1\}^n\}$$

where A is an m×n 0-1 matrix and 1 is the m-vector of 1's. Attempts at understanding the structure of this class of problems lead to the study of the set covering polytope

$$P_I(A) := \mathrm{conv}\{x \in \{0,1\}^n \mid Ax \geq 1\}$$

and its relation to the relaxed polytope

$$P(A) := \{x \in \mathbf{R}^n \mid Ax \geq 1,\ 0 \leq x \leq 1\}.$$

In a first paper on the facial structure of $P_I(A)$ [1], we described the facets of $P_I(A)$ defined by inequalities of the form $\alpha x \geq 2$, with $\alpha_j \in \{0,1,2\}$. More generally, we characterized this class of inequalities as obtainable by a particularly simple version of Chvatal's procedure [3] and established a connection between facet defining inequalities and full circulant submatrices C_k^{k-1} of A with $k \geq 3$.

In this paper, we connect the results of [1] to the theory of facet lifting [2, 4-7]. While in [1] our analysis centered on the role of certain subsets of the rows of A, here our focus is on subsets of columns of A. In particular, we start with a triplet of columns, plus some extra columns needed to make the resulting problem feasible, and consider a class of valid inequalities for the set covering polytope in this lower dimensional space (Section 2). We then lift the inequality into $\mathbf{R}^n$ by a specialized procedure that gives closed form expressions for the values of the coefficients (Section 3). Finally, we give necessary and sufficient conditions for a given inequality $\alpha x \geq 2$ to be obtainable by this procedure, and describe a modified lifting procedure, using a subset of the rows of A, that yields those inequalities not obtainable by the first procedure.

2. Inequalities from Circulants of Order 3

We first give some preliminary results on inequalities induced by submatrices of the form

$$C_3^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Let N and M denote the column and row index sets, respectively of A. We note that $P_I(A)$ is full dimensional if and only if every row of A has at least two 1's. More generally, the dimension of $P_I(A)$ is n-k, where k is the cardinality of a maximal set of rows that have a single 1, no two of which have their 1's in the same column.

<u>*Theorem 2.1*</u>. *Let* $N_1 \subseteq N$, $|N_1| = 3$, $N_2 = N \setminus N_1$, *and*

$$M_0 = \{i \in M \mid a_{ij} = 0, \forall j \in N_2\}.$$

The inequality

(2.1) $\quad\quad \Sigma(x_j : j \in N_1) \geq 2$

is valid for $P_I(A)$ *if and only if every column of* $A^{N_1}_{M_0}$ *contains a zero.*

A valid inequality (2.1) is minimal if and only if $A^{N_1}_{M_0}$ *does not have a zero row or two unequal rows with a single* 1 *in each.*

A valid inequality (2.1) for $P_I(A)$ *cuts off a vertex of P(A) if and only if every row of* $A^{N_1}_{M_0}$ *contains at most one zero.*

Finally, a valid inequality (2.1) defines a facet of $P_I(A)$ *if and only if every row of* $A^{N_1}_{M_0}$ *contains at most one* 0, *and* $A_{M\setminus M_0}$ *does not contain a 3xn submatrix of the form* (I_3, E) *(up to row permutations) where* I_3 *is the identity matrix with columns indexed by* N_1 *and E has a single column of* 1*'s with all other entries equal to* 0.

Proof: If there exists a column j_0 of $A^{N_1}_{M_0}$ which has no 0, then x^* defined by $x^*_j = 1$ if $j \in N_2 \cup \{j_0\}$, $x^*_j = 0$ otherwise, satisfies $Ax \geq 1$ and violates (2.1), hence (2.1) is not valid. If every column of $A^{N_1}_{M_0}$ has a zero then it takes at least two columns to cover M_0, hence (2.1) is valid.

If $A^N_{M_0}1$ contains a zero row or two unequal rows with a single 1 in each, say $a_{i_1j_1} = 1$, $a_{i_2j_2} = 1$, then (2.1) is not minimal, since in the first case every inequality is trivially valid ($P_I(A)$ is empty), and in the second case adding up rows i_1 and i_2 gives $x_{j_1} + x_{j_2} \geq 2$, which strictly dominates (2.1).

Conversely, suppose there is no zero row or two unequal rows with a single 1 in $A^N_{M_0}1$. Then reducing any of the coefficients of (2.1) invalidates the inequality. Indeed, if the coefficient of some x_j, $j \in N_1$, is replaced by $\alpha < 1$, each of the inequalities $x_{j_1} + x_{j_2} + \alpha x_{j_3} \geq 2$ and $x_{j_1} + \alpha x_{j_2} + x_{j_3} \geq 2$ cuts off the solution x^* defined by $x^*_j = 1$ for $j \in N_2 \cup \{j_2,j_3\}$, $x^*_j = 0$ otherwise; and the inequality $\alpha x_{j_1} + x_{j_2} + x_{j_3} \geq 2$ cuts off $\bar{x}$ defined by $\bar{x}_j = 1$, $j \in N_2 \cup \{j_1,j_2\}$. On the other hand, if the coefficient of some x_{j_*}, $j_* \in N_2$, is replaced by $\beta < 0$, the inequality $x_{j_1} + x_{j_2} + x_{j_3} + \beta x_{j_*} \geq 2$ cuts off both x^* and $\bar{x}$.

Next, let (2.1) be a minimal valid inequality for $P_I(A)$. If every row of $A^N_{M_0}1$ contains at most one 0, then x^* defined by $x^*_j = 1/2$, $j \in N_1$, $x^*_j = 1$, $j \in N_2$ is a vertex of $P(A)$ cut off by (2.1). If, on the other hand, $A^N_{M_0}1$ has a row with two 0's, then the corresponding inequality of $Ax \geq 1$ is of the form $x_{j_1} \geq 1$ for some $j_1 \in N_1$; and since column j_1 of $A^N_{M_0}1$ has at least one 0, there also exists an inequality of $Ax \geq 1$ of the form $x_{j_2} + x_{j_3} \geq 1$, where $\{j_1,j_2,j_3\} = N_1$. But then (2.1) is the sum of these two inequalities and cannot cut off any point of $P(A)$.

From this last statement it also follows that if $A^N_{M_0}1$ has a row with more than one 0, then (2.1) cannot be facet defining for $P_I(A)$. Also, if $A_{M \setminus M_0}$ contains a submatrix of the form (I_3, E) and k is the index of the column of all 1's of E, then every cover x^* satisfying (2.1) with equality also satisfies $x^*_k = 1$; hence (2.1) does not define a facet of $P_I(A)$.

Conversely, suppose $A^N_{M_0}1$ has at most one 0 in every row and $A_{M \setminus M_0}$ contains no submatrix of the form (I_3, E), with the columns of I_3 indexed by N_1. If $A_{M\setminus M_0}$ contains any rows with a single 1, and p is the maximum number of such rows whose 1's occur in distinct columns, then the dimension of $P_I(A)$ is $d = n - p$. We will show that (2.1) is facet defining by exhibiting d affinely independent points of $P_I(A)$ that satisfy (2.1) with equality.

For $k \in N_2$, define

$$T(k)_0 := \{\ i \in M \setminus M_0 \mid a_{ij} = 0,\ \forall j \in N \setminus \{k\}\ \}$$

and

$$T(k)_1 := \{\ i \in M \setminus M_0 \mid a_{ij} = 0,\ \forall j \in N_2 \setminus \{k\},\ \Sigma(a_{ij} : j \in N_1) = 1\ \}.$$

Clearly, the number of $k \in N_2$ such that $T(k)_0 \neq \emptyset$ is $p = n - d$. Let $e = (1, \ldots, 1)$, $e \in \mathbf{R}^n$, and let e_j be the j-th unit vector in $\mathbf{R}^n$. For every $k \in N$ such that if $k \in N_2$, $T(k)_0 = \emptyset$, define the n-vector

$$x^k = \begin{cases} e - e_k & k \in N_1 \\ e - e_{j(k)} & k \in N_2 : T(k)_0 = \emptyset, \end{cases}$$

where $j(k) \in N_1$ is chosen such that, if $\{j_1, j_2\} = N_1 \setminus j(k)$, $a_{ij_1} + a_{ij_2} = 1$ for all $i \in T(k)_1$ (if $T(k)_1 = \emptyset$, $j(k)$ is chosen arbitrarily). The existence of such $j(k)$ follows from the fact that $A_{M\setminus M_0}$ does not contain a submatrix of the form (I_3, E) with I_3 indexed by N_1.

The d vectors x^k are in $P_I(A)$, satisfy (2.1) with equality, and are affinely independent; hence (2.1) defines a facet of $P_I(A)$.||

Corollary 2.2 *If (2.1) is a minimal valid inequality for $P_I(A)$ and is not the sum of two inequalities of $A_{M_0}^{N_1}x \geq 1$, then $A_{M_0}^{N_1}$ contains a C_3^2.*

Proof. Let (2.1) be a minimal valid inequality for $P_I(A)$. If (2.1) is not the sum of two inequalities, then every row of $A_{M_0}^{N_1}$ has at least two 1's; also, every column of $A_{M_0}^{N_1}$ has a 0. Therefore, $A_{M_0}^{N_1}$ contains a C_3^2.||

We illustrate Theorem 2.1 on the following

Example 2.1 Let A be the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Setting $N_1 = \{1,2,3\}$ and $N_2 = \{4,5\}$, we have $M_0 = \{1,2,3\}$ and every column of $A^{N_1}_{M_0}$ contains a zero. Thus $x_1 + x_2 + x_3 \geq 2$ is a valid inequality for $P_I(A)$. It is also minimal, since every row of $A^{N_1}_{M_0}$ does not contain a zero row or a row with a single 1; and it cuts off the vertex $x = (1/2, \ldots, 1/2)$ of $P(A)$. It contains C_3^2 in its first three rows and columns. However it is not facet defining, since rows 6, 4, 7 form the matrix (I_3, E), with 1's in column 5; hence $x_5 = 1$ for every $x \in P_I(A)$ such that $x_1 + x_2 + x_3 = 2$. On the other hand, if any 0 in rows 4, 6, 7, is replaced by a 1, then (2.1) is facet defining for $P_I(A)$.||

3. The Family of Sequentially Lifted Inequalities

We now discuss the lifting of the inequalities introduced in Theorem 2.1.

<u>*Theorem 3.1*</u> *Let N_1, $N_2 \subseteq N$, $N_1 \cap N_2 = \emptyset$, $|N_1| = 3$, and let (2.1) be a minimal valid inequality for $P_I(A^{N_1 \cup N_2}) \neq \emptyset$.*

Let $j_1, \ldots, j_q$ be an arbitrary ordering of $N_3 = N \setminus (N_1 \cup N_2)$, where $q = |N_3|$, and define α_j, $j \in N_3$, recursively as follows:

For $\ell = 1, \ldots, q$, denote

$$J^{\ell} := \{j_1, \ldots, j_{\ell-1}\}, \text{ with } J^1 = \emptyset;$$

$$J_t^{\ell} := \{j \in J^{\ell} \mid \alpha_j = t\}, \; t = 0, 1, 2.$$

$$M(J_0^{\ell}) := \{i \in M \mid a_{ij} = 0, \forall j \in N_2 \cup J_0^{\ell}\}.$$

and define the conditions

(cl) $\quad a_{ij_\ell} = 1$ *for all* $i \in M(J_0^{\ell})$;

and

(c2) *there exists* $k \in N_1 \cup J_1^{\ell}$ *such that*

$$a_{ik} + a_{ij_\ell} \geq 1 \text{ for all } i \in M(J_0^{\ell}).$$

Set

$$\alpha_{j\ell} = \begin{cases} 2 & \text{if (c1) holds} \\ 1 & \text{if (c2) holds but (c1) does not} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\Sigma(x_j : j \in N_1) + \Sigma(\alpha_{j_\ell} x_{j_\ell} : \ell = 1, \ldots, q) \tag{3.2}$$

is a minimal valid inequality for $P_I(A)$. *Further, if (2.1) cuts off a fractional vertex of* $P(A^{N_1 \cup N_2})$, *then (3.2) cuts off a vertex of P(A).*

Finally if (2.1) defines a facet of $P_I(A^{N_1 \cup N_2})$, *then (3.2) defines a facet of* $P_I(A)$.

Before proving this theorem, we illustrate it.

Example 3.1. Let A be the matrix

$$\begin{pmatrix} 1&1&1&0&0&0&0&0&1 \\ 0&1&1&0&0&1&0&1&1 \\ 1&0&1&0&0&0&1&1&0 \\ 1&1&0&0&0&0&0&1&0 \\ 0&0&0&1&1&1&1&0&1 \\ 1&1&0&1&0&1&1&0&0 \\ 0&1&1&0&0&0&1&1&0 \end{pmatrix}$$

Let $N_1 = \{1,2,3\}$, $N_2 = \{4,5\}$. Then $M_0 = \{1,2,3,4,7\}$, and $x_1 + x_2 + x_3 \geq 2$ is a minimal valid inequality for $P_I(A^{N_1 \cup N_2})$, that cuts off the fractional vertex $x = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0, 1\right)$. Further, $P_I(A^{N_1 \cup N_2})$ is full dimensional and thus $x_1 + x_2 + x_3 \geq 2$ is facet defining. To lift this inequality, consider the ordering of N_3 given by (6,7,8,9).

For $j_1 = 6$, $J^1 = \emptyset$ and $M(J_0^1) = M_0$. Since neither (c1) nor (c2) holds, we set $\alpha_{j_1} = \alpha_6 = 0$.

For $j_2 = 7$, $J_0^2 = \{6\}$, $J_1^2 = \emptyset$, and $M(J_0^2) = \{1,3,4,7\}$. Condition (c1) does not hold but (c2) holds since $a_{i2} + a_{i7} \geq 1$ for all $i \in M(J_0^2)$. Thus we set

$\alpha_{j_2} = \alpha_7 = 1$.

For $j_3 = 8$, $J_0^3 = \{6\}$, $J_1^3 = \{7\}$ and $M(J_0^3) = \{1,3,4,7\}$. Again, (c1) does not hold but (c2) holds since $a_{i2}+a_{i8} \geq 1$ for all $i \in M(J_0^3)$, and so we set $\alpha_{j_3} = \alpha_8 = 1$.

Finally, for $j_4 = 9$, $J_0^4 = \{6\}$, $J_1^4 = \{7,8\}$, $M(J_0^4) = \{1,3,4,7\}$; (c1) does not hold but (c2) holds, since $a_{i2}+ a_{i9} \geq 1$ for all $i \in M(J_0^4)$. Thus we set $\alpha_{j_4} = \alpha_9 = 1$, and the lifted inequality is

(3.3) $$x_1 + x_2 + x_3 + x_7 + x_8 + x_9 \geq 2$$

According to the Theorem, (3.3) is valid and minimal for $P_I(A)$. Also, (3.3) cuts off a fractional vertex of $P(A)$, for instance $x = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0, 1, 0, 0, 0, 0\right)$; and it defines a facet of $P_I(A)$.

On the other hand, if we order N_3 as $\{6,7,9,8\}$, then $\alpha_9 = 2$ and $\alpha_8 = 0$, i.e. the lifted inequality is

$$x_1 + x_2 + x_3 + x_7 + 2x_9 \geq 2,$$

which is also minimal and facet defining for $P_I(A)$.

We will prove Theorem 3.1 by the technique known as *sequential lifting with complementing* (see [2, 6])

We start by complementing the variables x_j, $j \in N_1 \cup N_2$, i.e. by restating the problem in terms of a new set of variables y_j, $j \in N$, defined by

(3.4) $$y_j = \begin{cases} 1 - x_j & \text{if } j \in N_1 \cup N_2 \\ x_j & \text{if } j \in N_3 \end{cases}$$

Clearly, $x \in \{0,1\}^n$ satisfies $Ax \geq 1$ if and only if $y \in \{0,1\}^n$ satisfies

(3.5) $$\Sigma(a_{ij}y_j : j \in N_1 \cup N_2) - \Sigma(a_{ij}y_j : j \in N_3)$$
$$\leq \Sigma(a_{ij} : j \in N_1 \cup N_2) - 1 \qquad \forall i \in M$$

and $x \in \{0,1\}^n$ satisfies (3.2) if and only if $y \in \{0,1\}^n$ satisfies

(3.6) $$\Sigma(y_j : j \in N_1) + \Sigma(\beta_j y_j : j \in N_3) \leq 1 \ (= |N_1| - 2),$$

where $\beta_j = -\alpha_j$, $j \in N_3$. Define

$$P_I^C(A) := \text{conv}\{y \in \{0,1\}^n \mid y \text{ satisfies } (3.5)\}$$

We note the obvious facts that

(i) $x \in P_I(A)$ if and only if $y \in P_I^C(A)$ for y defined by (3.1);

(ii) $\dim P_I(A) = \dim P_I^C(A)$;

(iii) (3.2) is a (minimal) valid inequality for $P_I(A)$ if and only if (3.6) is a (maximal) valid inequality for $P_I^C(A)$;

(iv) (3.2) defines a facet of $P_I(A)$ if and only if (3.6) defines a facet of $P_I^C(A)$.

Since properties (i) - (iv) also hold for arbitrary submatrices of A, we can prove that (3.2) has the properties claimed in Theorem 3.1 with respect to $P_I(A)$ by showing that (3.6) has those properties with respect to $P_I^C(A)$.

Lemma 3.2 *Let*

(3.7) $$\Sigma(y_j : j \in N_1) \le 1$$

be a maximal valid inequality for $P_I^C(A^{N_1 \cup N_2}) \neq \emptyset$. *Let* $j_1, \ldots, j_q$ *be any ordering of* N_3 (*with* $q = |N_3|$). *Then*

(3.8) $$\Sigma(y_j : j \in N_1) + \Sigma(\beta_{j_\ell} y_{j_\ell} : \ell = 1, \ldots, q) \le 1$$

is a maximal valid inequality for $P_I^C(A)$, *with the coefficients* β_{j_ℓ} *defined recursively by* $\beta_{j_\ell} = 1 - z_{j_\ell}$, *where*

$$z_{j_\ell} = \max \Sigma(y_j : j \in N_1) + \Sigma(\beta_{j_k} y_{j_k} : k = 1, \ldots, \ell-1)$$
$$\text{s.t.}$$

$(\mathrm{IP})_\ell$ $$\Sigma(a_{ij} y_j : j \in N_1 \cup N_2) - \Sigma(a_{ij_k} y_{j_k} : k = 1, \ldots, \ell-1)$$
$$\le \Sigma(a_{ij} : j \in N_1 \cup N_2) - 1 + a_{ij_\ell}, \quad \forall\, i \in M$$
$$y_j \in \{0,1\},\ j \in N.$$

Further, if (3.7) cuts off a vertex of $P^C(A^{N_1 \cup N_2})$, *then (3.8) cuts off a vertex of* $P^C(A)$. *Finally if (3.7) defines a facet of* $P_I^C(A^{N_1 \cup N_2})$, *then (3.8) defines a facet of* $P_I^C(A)$.

Proof. If (3.8) is not valid, it cuts off some $y^* \in P_I^C(A)$. Let ℓ be the largest integer such that $y^*_{j_\ell} = 1$. Then

$$\Sigma(y^*_j : j \in N_1) + \Sigma(\beta_{j_k} y^*_{j_k} : k = 1, \ldots, \ell-1) + \beta_{j_\ell} > 1$$

which implies $\beta_{j_\ell} > 1 - z_{j_\ell}$, contradicting the definition of β_{j_ℓ}.

If (3.8) is valid but not maximal, then at least one of its coefficients β_j, $j \in N_3$, is not maximal, i.e. can be increased without cutting off any point of $P^C_I(A)$. Let ℓ be the smallest integer such that β_{j_ℓ} is not maximal, and let $\bar{y}$ be the solution to $(IP)_\ell$ that yielded the value z_{j_ℓ}. Then $(\bar{y}, 1, 0, \ldots, 0) \in P^C_I(A)$, where 1 is the component indexed by j_ℓ, and the 0's are those indexed by $j_{\ell+1}, \ldots, j_q$. Since β_{j_ℓ} is not maximal,

$$\Sigma(\bar{y}_j : j \in N_1) + \Sigma(\beta_{j_k} \bar{y}_k : k = 1, \ldots, \ell-1) + \beta_{j_\ell} < 1$$

or $\beta_{j_\ell} < 1 - z_{j_\ell}$, again contrary to the definition of β_{j_ℓ}.

Thus (3.8) is a maximal valid inequality for $P^C_I(A)$. If (3.7) cuts off a vertex $\hat{y}$ of $P^C_I(A^{N_1 \cup N_2})$, (3.8) obviously cuts off the vertex of $P^C_I(A)$ obtained from $\hat{y}$ by adding 0 components for all j_ℓ, $\ell = 1, \ldots, q$.

Now suppose (3.7) defines a facet of $P^C_I(A^{N_1 \cup N_2})$, and denote by A^q the matrix A whose columns are indexed by $N_1 \cup N_2 \cup \{j_1, \ldots, j_q\}$. We use induction on q. Since for $q = 0$ we have $P^C_I(A^0) = P^C_I(A^{N_1 \cup N_2})$, assume that (3.8) defines a facet of $P^C_I(A^q)$ for $q = 0, 1, \ldots, k$, and let $q = k + 1 \geq 1$. Denote by d the dimension of $P^C_I(A^k)$, and assume first that the dimension of $P^C_I(A^{k+1})$ is $d + 1$. Then there are d affinely independent points $y^i \in P^C_I(A^k)$, $i=1, \ldots, d$, each having $\gamma := |N_1 \cup N_2| + k$ components, that satisfy (3.8) (for $q = k$) with equality. Adding to each y^i a $(\gamma + 1)$-st component equal to 0 and defining the additional $(\gamma + 1)$-vector $y^{d+1} := (0, \ldots, 0, 1)$, we obtain $d + 1$ affinely independent points $y^i \in P^C_I(A^{k+1})$ that satisfy (3.8) (for $q = k + 1$) with equality.

Assume now that the dimension of $P^C_I(A^{k+1})$ is $d + r$ for some integer r, $2 \leq r \leq \gamma + 1 - d$, i.e. that the addition of column j_{k+1} increases the rank of the system by more than one. Then the system defining $P^C_I(A^k)$ has $r - 1$ inequalities of the form $y_{t_p} \leq 0$, $t_p \in N_2$, $p = 1, \ldots, r - 1$, with all t_p

distinct, such that the corresponding inequalities of $P_I^C(A^{k+1})$ are $y_{t_p} - y_{j_{k+1}} \leq 0$, $p = 1, \ldots, r-1$. To the set of $d+1$ affinely independent $(\gamma+1)$-vectors y^i constructed above we then add $r-1$ new $(\gamma+1)$-vectors of the form $y^{d+1+p} = (0, \ldots, 0, 1, 0, \ldots, 0, 1)$, $p = 1, \ldots, r-1$, with the first 1 in position t_p and the second in position j_{k+1}. Since the first $d+1$ vectors y^i all had $y^i_{t_p} = 0$ for $p = 1, \ldots, r-1$ (otherwise they would not have been in $P_I^C(A^k)$), the addition of each y^{d+1+p}, $p = 1, \ldots, r-1$ increases the rank of the resulting system by one; and thus we obtain a system of $d+r$ affinely independent points $y^i \epsilon P_I^C(A^{k+1})$, each of which satisfies (3.8) (for $q = k+1$) with equality.

This completes the induction and proves that (3.8) defines a facet of $P_I^C(A)$.||

From Lemma 3.2 it would seem that calculating the coefficients of a lifted inequality requires the solution of an integer program in $|N_1 \cup N_2| + \ell - 1$ variables for each coefficient β_{j_ℓ}, a task that is NP-complete. This is indeed the case when the problem at hand is an integer program with no special structure. In the case of the family of inequalities (3.2) for the set covering polytope, however, the structure of the problem allows one to solve this sequence of integer programs by a closed form expression, as we shall presently show. But first we need two more auxiliary results.

Lemma 3.3. For any ordering of N_3, we have $1 \leq z_j \leq 3$ and $0 \geq \beta_j \geq -2$ for all $j \in N_3$.

Proof. Since (3.7) is a maximal valid inequality for $P_I^C(A^{N_1 \cup N_2})$, the integer program $(IP)_\ell$ always has a feasible solution of the form $y_k = 1$ for some $k \epsilon N_1$, $y_i = 0$, $i \neq k$. Thus $z_j \geq 1$ and hence $\beta_j \leq 0$, $j = j_1, \ldots, j_q$. On the other hand, since $\beta_j \leq 0$ for all $j \epsilon N_3$, 3 is the unconstrained maximum of z_{j_ℓ} for all $\ell \epsilon \{1, \ldots, q\}$; hence $z_j \leq 3$ and $\beta_j \geq -2$, $j = j_1, \ldots, j_q$.||

Lemma 3.4. Let (3.7) be a maximal valid inequality for $P_I^C(A^{N_1 \cup N_2})$, and let $j_1, \ldots, j_q$ be an arbitrary ordering of N_3. Suppose β_j has been determined for $j = j_1 \ldots, j_{\ell-1}$, and let

$$J^{\ell} = \{j_1, \ldots, j_{\ell-1}\} \quad (J^{\ell} = \emptyset \text{ if } \ell = 1),$$
$$J_t^{\ell} = \{j \in J^{\ell} \mid \beta_j = -t\}, \; t = 0, 1, 2.$$

Then the value of z_{j_ℓ} *in* $(IP)_\ell$ *is the same as in*

$$z_{j_\ell} = \max \Sigma(y_j : j \in N_1) - \Sigma(y_j : j \in J_1^{\ell})$$

s.t.

$$(SIP)_\ell \qquad \Sigma(a_{ij}y_j : j \in N_1) - \Sigma(a_{ij}y_j : j \in J_1^{\ell})$$
$$\leq \Sigma(a_{ij} : j \in N_1) - 1 + a_{ij_\ell}, \; \forall i \in M(J_0^{\ell})$$
$$\Sigma(y_j : j \in J_1^{\ell}) \leq 1$$
$$y_j \in \{0,1\}, \; j \in N_1 \cup J_1^{\ell}.$$

Proof. Since the objective function coefficient of y_j is 0 for $j \in N_2 \cup J_0^{\ell}$, while the constraint coefficients of y_j are all nonnegative for $j \in N_2$ and all nonpositive for $j \in J_0^{\ell}$, we can set $y_j = 0$ for $j \in N_2$ and $y_j = 1$ for $j \in J_0^{\ell}$ without affecting the value of z_{j_ℓ}. This amounts to removing the variables y_j, $j \in N_2 \cup J_0^{\ell}$, from $(IP)_\ell$, and adding $\Sigma(a_{ij} : j \in J_0^{\ell})$ to the right-hand side of the i-th constraint for $i \in M$. Now for $M(J_0^{\ell})$, $\Sigma(a_{ij} : j \in N_2 \cup J_0^{\ell}) = 0$ by definition, and so the right hand side coefficient is $\Sigma(a_{ij} : j \in N_1) - 1 + a_{ij_\ell}$, as claimed for $(SIP)_\ell$.

Further, these value assignments for y_j, $j \in N_2 \cup J_0^{\ell}$ make the constraints $i \in M \setminus M(J_0^{\ell})$ redundant, since

$$\Sigma(a_{ij}y_j : j \in N_1 \cup N_2) - \Sigma(a_{ij}y_j : j \in J^{\ell})$$
$$= \Sigma(a_{ij}y_j : j \in N_1) - \Sigma(a_{ij}y_j : j \in J^{\ell})$$
$$\leq \Sigma(a_{ij} : j \in N_1)$$
$$\leq \Sigma(a_{ij} : j \in N_1 \cup N_2 \cup J_0^{\ell}) - 1 + a_{ij_\ell}$$

where the last inequality follows from the fact that

$$\Sigma(a_{ij} : j \in N_2 \cup J_0^{\ell}) \geq 1$$

for $i \in M \setminus M(J_0^{\ell})$ (by definition of $M(J_0^{\ell})$).

Next, the maximum amount by which $\Sigma(y_j : j \in N_1)$ could possibly be increased by setting $y_j = 1$ for some or all $j \in J_1^{\ell} \cup J_2^{\ell}$ is 2 (from 1 to 3); hence we can set $y_j = 0$ for all $j \in J_2^{\ell}$ (as $\beta_j = -2$), and we can impose the constraint $\Sigma(y_j : j \in J_1^{\ell}) \leq 1$ (as $\beta_j = -1$ for all $j \in J_1^{\ell}$) without affecting the value of z_{j_ℓ}.||

Proof of Theorem 3.1. From Lemma 3.2, Theorem 3.1 is true if and only if $\alpha_{j_\ell} = -\beta_{j_\ell}$, $\ell = 1, \ldots, q$, where the α_{j_ℓ} are defined by (3.1) and the β_{j_ℓ} by $(SIP)_\ell$.

Note that the last inequality of $(SIP)_\ell$ can be written as $\Sigma(y_j : j \in J_1) =$ 0 or 1. Thus $z_{j_\ell} = \max\{z^0_{j_\ell}, z^1_{j_\ell}\}$ where $z^0_{j_\ell}$, $z^1_{j_\ell}$ denote the maximum in $(SIP)_\ell$ when $\Sigma(y_j : j \in J_1) = 0$ and 1, respectively.

Consider the conditions of Theorem 3.1 defining α_{j_ℓ}. If (c1) holds, then for every $i \in M(J_0^{\ell})$ the right hand side of the corresponding constraint of $(SIP)_\ell$ is equal to $\Sigma(a_{ij} : j \in N_1)$, hence $z_{j_\ell} = z^0_{j_\ell} = 3$ (and $\Sigma(y_j : j \in J_1^{\ell}) =$ 0). Conversely, if $z_{j_\ell} = 3$, i.e., $y_j = 1$ $\forall j \in N_1$, the right hand side of every inequality $i \in M(J_0^{\ell})$ must be equal to $\Sigma(a_{ij} : j \in N_1)$, hence (c1) must hold. Thus $z_{j_\ell} = 3$ if and only if (c1) holds.

If (c1) does not hold, then $z_{j_\ell} \leq 2$. Now suppose (c2) holds, i.e. there exists $k \in N_1 \cup J_1^{\ell}$ such that $a_{ik} + a_{ij_\ell} \geq 1$ for all $i \in M(J_0^{\ell})$. If $k \in N_1$, then $\bar{y}$ defined by $\bar{y}_j = 1$, $j \in N_1 \setminus \{k\}$, $\bar{y}_j = 0$ otherwise is feasible in $(SIP)_\ell$, since

$$\Sigma\, a_{ij}\bar{y}_j : j \in N_1) - \Sigma(a_{ij}\bar{y}_j : j \in J_1^{\ell}) = \Sigma(a_{ij} : j \in N_1 \setminus \{k\})$$
$$\leq \Sigma(a_{ij} : j \in N_1) - 1 + a_{ij_\ell}.$$

If, on the other hand, $k \in J_1^{\ell}$, then $\hat{y}$ defined by $\hat{y}_j = 1$, $j \in N_1 \cup \{k\}$, $\bar{y}_j = 0$ otherwise, is feasible since

$$\Sigma(a_{ij}\hat{y}_j : j \in N_1) - \Sigma(a_{ij}\hat{y}_j : j \in J_1^{\ell}) = \Sigma(a_{ij} : j \in N_1) - a_{ik}$$
$$\leq \Sigma(a_{ij} : j \in N_1) - 1 + a_{ij_\ell}$$

Further, for both $\bar{y}$ and $\hat{y}$, $z_{j_\ell} = 2$.

Finally, assume that neither (c1) nor (c2) holds. Then for each $k \in N_1 \cup J_1^{\ell}$ there exists some $i \in M(J_0^{\ell})$ such that $a_{ik} = a_{ij_\ell} = 0$. We consider two cases.

Case 1: $\Sigma(y_j : j \in J_1^{\ell}) = 0$. Since $z_{j_\ell} \leq 2$, at most two of the variables y_j, $j \in N_1$, can be equal to 1 in any solution. Let $y_k = 0$, and let $i(k) \in M(J_0^{\ell})$ be such that $a_{i(k)j_\ell} = 0$. then

$$\Sigma(a_{i(k)j}y_j : j \in N_1\backslash\{k\}) \leq \Sigma(a_{i(k)j} : j \in N_1\backslash\{k\}) - 1,$$

i.e. at most one of the two variables y_j, $j \in N_1 - \{k\}$, can be equal to 1. Hence $z_{j_\ell} = 1$.

Case 2: $\Sigma(y_j : j \in J_1^{\ell}) = 1$. Let $y_k = 1$, $y_j = 0$, $j \in J_1^{\ell} \setminus \{k\}$, and let $i(k) \in M(J_0^{\ell})$ be such that $a_{i(k)k} = a_{i(k)j_\ell} = 0$. Then

$$\Sigma(a_{i(k)j}y_j : j \in N_1) - \Sigma(a_{i(k)j}y_j : j \in J_1^{\ell}) = \Sigma(a_{i(k)j}y_j : j \in N_1) - a_{i(k)k}$$
$$\leq \Sigma(a_{i(k)j} : j \in N_1) - 1$$

or, since $a_{i(k)k} = 0$,

$$\Sigma(a_{i(k)j}y_j : j \in N_1) \leq \Sigma(a_{i(k)j} : j \in N_1) - 1,$$

which means that at most two of the variables y_j, $j \in N_1$, can be equal to 1 in any solution to $(SIP)_\ell$. This, together with $y_k = 1$, implies $z_{j_\ell} = 1$.

We have shown that $\beta_{j_\ell} = 1 - z_{j_\ell} = -2$ if (c1) holds, $\beta_{j_\ell} = -1$ if not (c1) but (c2) holds, and $\beta_{j_\ell} = 0$ if neither (c1) nor (c2) is satisfied. Thus $\beta_{j_\ell} = -\alpha_{j_\ell}$, $\ell = 1, \ldots, q$ for α_{j_ℓ} defined by (3.1).||

Theorem 3.1 gives a sufficient condition for an inequality (3.2) to define a facet of $P_I(A)$. The condition, however, is not necessary. This is illustrated by the following.

Example 3.2. Let A be the matrix

$$\begin{pmatrix} 0&1&1&0&0&0&1&0&1 \\ 1&0&1&0&0&0&0&1&0 \\ 1&1&0&0&0&0&0&1&1 \\ 1&0&0&0&1&0&0&1&1 \\ 0&1&0&0&1&0&1&1&0 \\ 0&0&1&0&1&0&1&1&0 \\ 1&1&0&1&0&1&0&1&0 \end{pmatrix}$$

and let $N_1 = \{1,2,3\}$, $N_2 = \{4,5,6\}$. Then $x_1 + x_2 + x_3 \geq 2$ defines a minimal inequality for $P_I(A^{\{1,...,6\}})$, but not a facet, since $A^{\{1,...,6\}}$ contains (I_3, E) as a submatrix. However, for the ordering (7, 8, 9) of N_3 in the lifting procedure, we obtain the inequality $x_1 + x_2 + x_3 + x_7 + x_8 + x_9 \geq 2$, which defines a facet of $P_I(A)$.||

A given inequality (2.1) can give rise via lifting to many different inequalities (3.2), depending on the sequence in which the coefficients α_j are calculated. The earlier in the sequence a given coefficient is calculated, the lower its value (in the weak sense). To be precise, the coefficients α_j have the following property.

<u>*Corollary 3.5*</u>. *Let $\{j_1, \ldots, j_q\}$ be an arbitrary ordering of N_3, and for*

any ℓ, let $\alpha_{j_\ell}^{(k)}$ be the value of α_{j_ℓ} in the inequality (3.2) associated with the ordering obtained from $\{j_1, \ldots, j_q\}$ by moving j_ℓ to the k^{th} position. Then

$$\alpha_{j_\ell}^{(k)} \leq \alpha_{j_\ell}^{(k+1)}, \quad k = 1, \ldots, q-1.$$

Proof. Consider the problem $(SIP)_\ell$ used to calculate the value of $\alpha_{j_\ell}^{(k)}$ and $\alpha_{j_\ell}^{(k+1)}$ and denote by $z_{j_\ell}^{(k)}$ and $z_{j_\ell}^{(k+1)}$ the corresponding values of z_j. Then clearly the solution that yields the value of $z_{j_\ell}^{(k)}$ is also feasible to the problem whose optimum is $z_{j_\ell}^{(k+1)}$; hence $z_{j_\ell}^{(k)} \leq z_{j_\ell}^{(k+1)}$, i.e., $\alpha_j^{(k)} \leq \alpha_j^{(k+1)}$.||

Since the coefficients of (3.2) vary in size between 0 and 2, the question arises as to whether the range of variation of a given coefficient α_j

as a result of changes in its position in the sequence $\alpha_{j_1}, \ldots, \alpha_{j_q}$ can be narrowed down to less than 2, as in the case of a knapsack polytope, where this range is 1. The answer to this question is negative, as shown by the following example.

Example 3.4. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

with $N_1 = \{1,2,3\}$, $N_2 = \emptyset$. The lifting procedure yields the two inequalities

$$x_1 + x_2 + x_3 + 0.x_4 + 2x_5 \geq 2$$

$$x_1 + x_2 + x_3 + 2x_4 + 0.x_5 \geq 2$$

for the sequences 4, 5 and 5, 4, respectively, and each of the variables x_4, x_5 have coefficients that differ by 2 in the two inequalities.||

For any subset $S \subset M$, $P_I(A) \subseteq P_I(A_S)$ and thus any inequality valid for $P_I(A_S)$ is also valid for $P_I(A)$. Therefore we have

Corollary 3.6. Let S be a proper subset of M, $|S| \geq 3$, and let

$$(3.9) \qquad \Sigma(x_j : j \in N_1) + \Sigma(\alpha_j x_j : j \in N_3) \geq 2$$

be a lifted inequality obtained by applying Theorem 3.1 with $M(J_0^{\ell})$ replaced by

$$S(J_0^{\ell}) := \{i \in S \mid a_{ij} = 0, \forall j \in N_2 \cup J_0^{\ell}\}$$

in the conditions (c1) and (c2) defining α_{j_ℓ}. Then (3.9) is a valid inequality for $P_I(A)$.

We will use the device of working with a subset of the rows of A to prove the key result of Section 4. Here we illustrate one of the situations when this device is useful.

Example 3.3. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Let $N_1 = \{1,2,3\}$, $N_2 = \emptyset$. The inequality

$$x_1 + x_2 + x_3 \geq 2$$

is valid but not minimal for $P_I(A^{N_1 \cup N_2})$, since $P_I(A^{N_1 \cup N_2}) = \emptyset$ and any inequality is valid for an empty polytope. However, this is a minimal valid inequality for $P_I(A_S^{N_1 \cup N_2})$ with $S := \{2,3,4\}$, and when lifted via the procedure of Theorem 3.1 (with A replaced by A_S) it yields the inequality

$$x_1 + x_2 + x_3 + 2x_4 + x_5 + x_6 + x_7 \geq 2,$$

which is minimal and facet defining for both $P_I(A_S)$ and $P_I(A)$.||

Naturally, when $P_I(A^{N_1 \cup N_2}) \neq \emptyset$, the inequality (3.2) obtained by applying the lifting procedure to A dominates any inequality obtained by applying the same procedure to A_S for some $S \subset M$.

Note that although the lifting procedure in principle involves solving an integer program to calculate the value of each coefficient, Theorem 1 gives a closed form expression for the values of the coefficients, which makes it possible to calculate them efficiently. The work involved in calculating all the coefficients of an inequality (3.2) is $O(mn^2)$, where $m = |M|$ and $n = |N|$. Next we identify an important subclass of the class (3.2) of inequalities, whose members are independent of the sequence in which their coefficients are calculated, and can be obtained by work of $O(mn)$.

For any triplet $N_1 \subset N$ and any $N_2 \subseteq N \setminus N_1$, define, as before, $M_0 := \{i \in M \mid a_{ij} = 0, \forall j \in N_2\}$ and $N_3 := N \setminus (N_1 \cup N_2)$. We will say that N_2 is *N_1-maximal* if for each $j \in N_3$, there exists $k(j) \in N_1$ such that for all $i \in M_0$, $a_{ij} = 0$ implies $a_{ik(j)} = 1$. N_1-maximal sets N_2 with respect to a given triplet N_1 need not be unique.

Corollary 3.7. Let N_k, $k = 1, 2, 3$, be as in Theorem 3.1, let (2.1) be a *minimal valid inequality for* $P_I(A^{N_1 \cup N_2})$, *and let* N_2 *be* N_1*-maximal. Then for any ordering of* N_3, *the coefficients defined by (3.1) are given by*

$$(3.10) \qquad \alpha_j = \begin{cases} 2 \ \text{if } a_{ij} = 1 \text{ for all } i \in M_0 \\ 1 \ \text{otherwise.} \end{cases}$$

Proof: For $j \in N_3$ such that $a_{ij} = 1$ for all $i \in M_0$, condition (c1) of Theorem 3.1 holds and thus $\alpha_j = 2$. For all other $j \in N_3$, condition (c2) holds since N_2 is N_1-maximal, hence $\alpha_j = 1.\|$

Note that not only is the definition (3.10) of the coefficients α_j simpler than (3.1), but calculating these coefficients also involves less work: given some triplet $N_1 \subset N$, finding a set $N_2 \subseteq N \setminus N_1$, such that N_2 is N_1-maximal and calculating all the coefficients α_j defined by (3.10) require, $O(mn)$ work.

Besides requiring that N_2 be N_1-maximal, Corollary 3.7 also assumes that (2.1) is a minimal valid inequality for $P_I(A^{N_1 \cup N_2})$. This imposes further conditions on N_1 and N_2, without which the lifting procedure may break down; namely, that the submatrix $A^{N_1}_{M_0}$ (whose definition depends on N_2 via M_0) have no zero rows and no pair of unequal rows with a single 1 in each. An N_1-maximal set N_2 that satisfies these conditions will be called *admissible*. If, in addition, (2.1) is to cut off some vertex of $P(A)$, then $A^{N_1}_{M_0}$ must contain a C^2_3.

The following procedure identifies an admissible N_1-maximal set N_2 for a given triplet N_1 that contains C^2_3 as a submatrix. We assume that A has at least two 1's in every row.

Step 0. Set $N_2 := \emptyset$, $N_3 := N \setminus N_1$. Choose any $j_* \in N_1$, set $M_0 := \{i \in M \mid \Sigma(a_{ij} : j \in N_1) \geq 2 \text{ or } a_{ij_*} = 1\}$, $M_1 := M \setminus M_0$, and go to 1.

Step 1. If $N_3 = \emptyset$, stop.

If $N_3 \neq \emptyset$ but $M_1 = \emptyset$, choose $j \in N_3$, set $N_3 := N_3 \setminus \{j\}$ and go to 2.

If $N_3 \neq \emptyset$ and $M_1 \neq \emptyset$, choose $j \in N_3$ such that $a_{ij} = 1$ for some $i \in M_1$. If no such j exists, stop; otherwise set $N_3 := N_3 \setminus \{j\}$, and go to 2.

Step 2. If for each $k \in N_1$ there exists $i(k) \in M_0$ such that $a_{i(k)k} = a_{i(k)j} = 0$, set $N_2 := N_2 \cup \{j\}$, $M_0 := M_0 \setminus \{i \in M_0 \mid a_{ij} = 1\}$, and (if $M_1 \neq \emptyset$) $M_1 := M_1 \setminus \{i \in M_1 \mid a_{ij} = 1\}$. Go to 1.

When the procedure stops, $M_1 = \emptyset$ and N_2 is N_1-maximal and admissible. The procedure may stop with $N_2 = \emptyset$, which is trivially N_1-maximal and admissible but useless. In such a case, another choice of j_* and/or of the order in which the elements of N_3 are considered is likely to yield a different set N_2.

Example 3.4. Let A be the matrix

$$\begin{pmatrix} 0&1&1&1&1&1&0&0&0&0&1&0&0&0&1 \\ 1&0&1&1&1&0&1&0&0&0&1&1&0&0&0 \\ 1&1&0&1&1&0&0&1&0&0&0&1&1&0&0 \\ 1&1&1&0&1&0&0&0&1&0&0&0&1&1&0 \\ 1&1&1&1&0&0&0&0&0&1&0&0&0&1&1 \end{pmatrix}$$

For $N_1 = \{1,2,3\}$, the (unique) N_1-maximal (and admissible) set N_2 is $\{9,10,14\}$. Similarly, the N_1-maximal set N_2 happens to be unique (and admissible) for each of the $\binom{5}{2}$ triplets of the set $\{1, \ldots, 5\}$ (the index set of the first 5 columns). Table 1 shows for each N_1 the corresponding N_1-maximal admissible set N_2 and the coefficients defined by (3.10) for the associated inequality (3.9) (which is also the unique inequality (3.2) corresponding to the pair (N_1, N_2)).||

Table 1

N_1	N_2	α_j														
		j : 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1, 2, 3	9, 10, 14	1	1	1	2	2	1	1	1	0	0	1	1	1	0	1
1, 2, 4	8, 10	1	1	2	1	2	1	1	0	1	0	1	1	1	1	1
1, 2, 5	8, 9, 13	1	1	2	2	1	1	1	0	0	1	1	1	0	1	1
1, 3, 4	7, 10	1	2	1	1	2	1	0	1	1	0	1	1	1	1	1
1, 3, 5	7, 9	1	2	1	1	2	1	0	1	0	1	1	1	1	1	1
1, 4, 5	7, 8, 12	1	2	2	1	1	1	1	0	1	1	1	0	1	1	1
2, 3, 4	6, 10, 15	2	1	1	1	2	0	1	1	1	0	1	1	1	1	0
2, 3, 5	6, 9	2	1	1	2	1	0	1	1	0	1	1	1	1	1	1
2, 4, 5	6, 8	2	1	2	1	1	0	1	0	1	1	1	1	1	1	1
3, 4, 5	6, 7, 11	2	2	1	1	1	0	0	1	1	1	0	1	1	1	1

4. All the Facets with Coefficients in {0,1,2}

In this section we show that every inequality with coefficients in {0,1,2} that is facet inducing for $P_I(A)$ can be obtained by the lifting procedure of Theorem 3.1 from a minimal valid inequality to $P_I(A_R^{N_1 \cup N_2})$, for some triplet $N_1 \subseteq N$ and associated sets $N_2 \subseteq N$, $R \subseteq M$. For this we need some results of [1].

We have shown in [1] that all minimal inequalities of $P_I(A)$ with coefficients equal to 0, 1 or 2 can be generated by the following procedure applied to subsets S of M:

Procedure C

(i) Add the inequalities $a_{i1}x_1 + \ldots + a_{in}x_n \geq 1$ for all $i \in S$;

(ii) divide the resulting inequality by $|S| - \varepsilon$, where $0.5 < \varepsilon < 1$; and

(iii) round up all coefficients to the nearest integer.

We denote by $\alpha^S x \geq 2$ the valid inequality so obtained and by C the class of all such inequalities. Procedure C is a particular variant of Chvatal's well known, more general procedure [3].

The coefficients of the inequality resulting from procedure C are

$$\alpha_j = \begin{cases} 2 & \text{if } a_{ij} = 1, \ \forall i \in S \\ 0 & \text{if } a_{ij} = 0, \ \forall i \in S \\ 1 & \text{otherwise.} \end{cases}$$

We denote

$$J_k := \{j \in N \mid \alpha_j = k\}, \qquad k = 0, 1, 2$$

and

$$M(J_0) := \{i \in M \mid a_{ij} = 0, \forall j \in J_0\}.$$

For any submatrix A_S^H of A, a pair j, k ε H is called a *2-cover* of A_S^H if $a_{ij} + a_{ik} \geq 1$ for all i ε S. The *2-cover graph* of A_S^H has a vertex for every j ε H and an edge for every 2-cover of A_S^H.

The following result is from [1] (Corollary 2.5 and part of Theorem 2.6).

<u>*Lemma 4.1*</u>. *The inequality $\alpha^S x \geq 2$ is minimal if and only if the 2-cover graph of $A_{M(J_0)}^{J_1}$ has no isolated vertices. If $\alpha^S x \geq 2$ defines a facet of $P_I(A)$, then every component of the 2-cover graph of $A_{M(J_0)}^{J_1}$ has an odd cycle.*

The following is a key result of this section.

<u>*Theorem 4.2*</u>. *Let $\alpha x \geq 2$ be a facet defining inequality for $P_I(A)$, with $|M| \geq 4$. Then there exists a triplet $N_1 \subset J_1$ such that*

(2.1) $$\Sigma(x_j : j \in N_1) \geq 2$$

is a minimal valid inequality for $P_I(A^{N_1 \cup J_0})$. Further, the inequality $\alpha x \geq 2$ can be obtained from (2.1) by the lifting procedure of Theorem 3.1 if and only if the 2-cover graph of $A_{M(J_0)}^{J_1}$ is connected.

Proof. Since $\alpha x \geq 2$ is facet defining, it belongs to the class C. Hence $\alpha_j = 2$ if $a_{ij} = 1$ for all i ε $M(J_0)$, $\alpha_j = 0$ if $a_{ij} = 0$ for all i ε $M(J_0)$, and $\alpha_j = 1$ otherwise.

Let $G(J_1)$ denote the 2-cover graph of $A_{M(J_0)}^{J_1}$. Since $\alpha x \geq 2$ is facet defining, $G(J_1)$ has a path with at least three vertices (Lemma 4.1). Choosing for N_1 any three consecutive vertices of such a path guarantees that the

2-cover graph of $A^{N_1}_{M(J_0)}$ has no isolated vertex; hence that (2.1) is a minimal valid inequality for $P_I(A^{N_1 \cup J_0})$.

Now let $j_1, \ldots, j_q$ be any ordering of $N_3 := N \setminus (N_1 \cup J_0)$ (where $q = |N_3|$) such that for $k \in \{1, \ldots, q\}$, every $j_k \in J_1$ is adjacent in $G(J_1)$ to some $j \in N_1 \cup \{j_1, \ldots, j_{k-1}\}$. Such an ordering exists if and only if $G(J_1)$ is connected.

Suppose this is the case. For $j_k \in J_2$, since $a_{ij} = 1$ for all $i \in M(J_0)$, condition (c2) of Theorem 3.1 holds and thus j_k gets a coefficient of 2, which is the value of α_{j_k}. For $j_k \in J_1$, since there exists $j_* \in N_1 \cup \{j_1, \ldots, j_{k-1}\}$ such that $a_{ij_*} + a_{ij_k} \geq 1$ for all $i \in M(J_0)$, j_k gets a coefficient of 1, which again is the value of α_{j_k}.

Suppose now that $G(J_1)$ is not connected. Since N_1 is chosen from a path, hence from one component, there exists a component of $G(J_1)$ whose vertices do not contain any element of N_1 and are not adjacent in $G(J_1)$ to any element of N_1. Let J_{11} be the vertex set of this component. Then for any ordering of N_3, the first element from J_{11}, say j_k, will not form a 2-cover with any of the elements in $N_1 \cup \{j_1, \ldots, j_{k-1}\}$ and will therefore be assigned a coefficient of 0, which is different from the value of α_{j_k}. This implies that every inequality lifted from (2.1) by the procedure of Theorem 3.1 has a coefficient different from α_j for at least one $j \in J_{11} \subset J_1$, i.e. $\alpha x \geq 2$ cannot be obtained in this fashion.||

Theorem 4.2 should not surprise anyone familiar with lifting theory. Although if (2.1) is a minimal valid inequality for $P_I(A^{N_1 \cup J_0})$, then every minimal valid inequality for $P_I(A)$ which has coefficients identical to those of (2.1) for $j \in N_1 \cup J_0$ can be obtained from (2.1) by lifting [2, 7], the kind of lifting required may not be sequential, but simultaneous. What is specific to the class of inequalities discussed in this paper, however, is that in their case sequential lifting is sufficient for generating all of them, provided the procedure is extended to encompass restrictions of the row set M of A to some subset of M, in the vein of Corollary 3.6. To show how this can be done, we need another result of [1].

Given an inequality $\alpha^S x \geq 2$ in class C, a subset $T \subset M(J_0)$ is called *C-equivalent to S*, if $\alpha^T = \alpha^S$, i.e. T gives rise to the same inequality as S. T is a *minimal C-equivalent subset* of $M(J_0)$ if no proper subset of T is C-equivalent to $M(J_0)$.

For $k \geq 3$, we denote by C_k^{k-1} the complement of a permutation matrix, i.e. a square 0-1 matrix of order k, with exactly one 0 in every row and column.

<u>*Lemma 4.3*</u>. *(Theorem 3.1 of [1]) For every minimal C-equivalent subset T of $M(J_0)$, $A_T^{J_1}$ contains a submatrix C_t^{t-1}.*

Let $\alpha x \geq 2$ be a minimal valid inequality for $P_I(A)$, and let T be a minimal C-equivalent subset of $M(J_0)$. Let K be the column index set of a C_t^{t-1} contained in $A_T^{J_1}$, and define $L := K \cup J_0$, $R := T \cup (M \setminus M(J_0))$.

It is easy to see that any triplet $N_1 \subseteq K$ gives rise to a submatrix $A_T^{N_1}$ that has at least two 1's in every row and contains a C_3^2. To identify the latter, just take the three rows which contain a 0. An important property of $A_T^{J_1}$ is the following.

<u>*Lemma 4.4*</u>. *The 2-cover graph of $A_T^{J_1}$ is connected.*

Proof. Every pair of columns of C_t^{t-1} is a 2-cover of $A_T^{J_1}$, hence the column set K of C_t^{t-1} induces a clique in the 2-cover graph $G(J_1)$ of $A_T^{J_1}$. On the other hand, for every column $j \in J_1 \setminus K$ of $A_T^{J_1}$, there exists some column $k(j) \in K$ whose unique 0 occurs in a row i with $a_{ij} = 1$, i.e. such that j and $k(j)$ form a 2-cover $A_T^{J_1}$.||

We are now ready to state the main result of this section.

<u>*Theorem 4.5*</u>. *Let $|M| \geq 4$ and let $\alpha x \geq 2$ be a minimal valid inequality for $P_I(A)$. Then there exists a triplet $N_1 \subset N$ and a subset $T \subseteq M$ such that*

(2.1) $$\Sigma(x_j : j \in N_1) \geq 2$$

is a minimal valid inequality for $P_I(A_S^{N_1 \cup N_2})$, *where* $N_2 := J_0$ *and* $S := T \cup (M \setminus M(J_0))$. *Further,* $\alpha x \geq 2$ *can be obtained from* (2.1) *by the lifting procedure of Theorem 3.1, with* $M(J_0^{\ell})$ *replaced by*

$$S(J_0^{\ell}) := \{i \in S \mid a_{ij} = 0, \forall j \in N_2 \cup J_0^{\ell}\}$$

in the conditions (c1) and (c2) defining the coefficients $\alpha_{j\ell}$.

Proof. Since $\alpha x \geq 2$ is minimal, it belongs to the class C. From Lemmas 4.3 and 4.4, there exists a minimal C-equivalent subset T of $M(J_0)$ such that the 2-cover graph of $A_T^{J_1}$ is connected. Therefore there exists a triplet $N_1 \subset J_1$ such that (2.1) is a minimal valid inequality for $P_I(A_S^{N_1 \cup N_2})$, where $N_2 := J_0$ and $S := T \cup (M \setminus M(J_0))$. Since the 2-cover graph of $A_T^{J_1}$ is connected, from Theorem 4.2 $\alpha x \geq 2$ can be obtained by applying to (2.1) the lifting procedure of Theorem 3.1, with $M(\emptyset)$ $(= M_0)$ replaced by $S(\emptyset) = T$ and, more generally, $M(J_0^{\ell})$ replaced by $S(J_0^{\ell}) := \{i \in S \mid a_{ij} = 0, \forall j \in N_2 \cup J_0^{\ell}\}$.||

Example 4.1. *Consider the matrix*

$$A = \begin{pmatrix} 1&1&0&0&0&1&0&1&1 \\ 0&1&1&0&1&1&0&1&0 \\ 1&0&1&0&1&0&1&1&0 \\ 0&1&0&0&1&0&1&0&1 \\ 1&0&1&0&0&1&1&0&1 \\ 0&1&0&0&0&1&1&1&1 \\ 1&0&1&0&1&1&0&1&1 \\ 0&1&0&0&1&0&1&1&0 \\ 1&0&1&1&0&1&0&0&1 \\ 0&1&0&1&0&0&1&0&0 \end{pmatrix}$$

and the inequality

(4.1) $$x_1 + x_2 + x_3 + x_5 + x_6 + x_7 + x_8 + x_9 \geq 2$$

We have $J_0 = \{4\}$, $J_2 = \emptyset$, $J_1 = \{1,2,3,5,6,7,8,9\}$, and $M(J_0) = \{1, \ldots, 8\}$. We choose $N_1 = \{1,2,3\}$; then $x_1 + x_2 + x_3 \geq 2$ is a minimal valid inequality for $P_I(A^{N_1 \cup J_0}) = P_I(A^{\{1,\ldots,4\}})$, which is also facet defining. Since the 2-cover graph $G(J_1)$ of $A_{M(J_0)}^{J_1}$, shown in Figure 4.1, is disconnected, none of the lifted inequalities is (4.1). The coefficients of these inequalities are shown in Table 2.

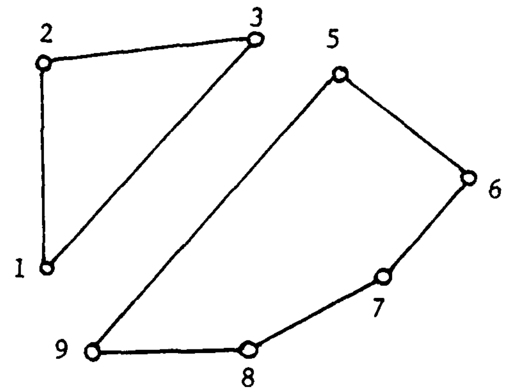

Figure 4.1

Table 2

Coefficients of lifted inequalities

No. \ j	1	2	3	4	5	6	7	8	9	Ordering of N_3
1	1	1	1	0	0	2	1	1	2	5, 6, 7, 8, 9
2	1	1	1	0	2	0	2	1	1	6, 5, 7, 8, 9
3	1	1	1	0	1	2	0	2	1	7, 5, 6, 8, 9
4	1	1	1	0	1	1	2	0	2	8, 5, 6, 7, 9
5	1	1	1	0	2	1	1	2	0	9, 5, 6, 7, 8

The orderings of N_3 considered are those in which the elements take turns to be chosen first, while the remaining elements keep their natural order. No other orderings yield different inequalities. Each of the 5 inequalities is facet defining for $P_I(A)$, as is inequality (4.1).

Now consider the row set $T = \{3,5,6,7\}$, which contains a C_4^3 with columns indexed by $K = \{3,6,7,8\}$. The 2-cover graph of $A_T^{J_1}$ is shown in Figure 2.

The inequality $x_1 + x_2 + x_3 \geq 2$ is minimal for $P_I(A_S^{N_1 \cup N_2})$, where $S = \{3,5,6,7,9,10\}$. Applying the lifting procedure with $M(\emptyset)$ replaced by

$T = \{3,6,7,8\}$, and any ordering of N_3 in which 5 does not precede all of 6, 7, 8, 9, yields the inequality (4.1).||

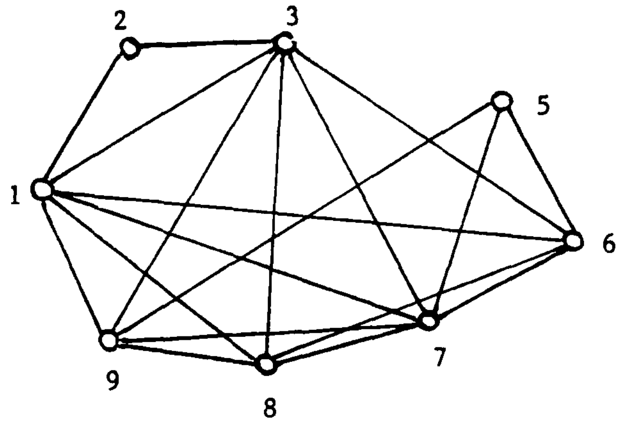

Figure 4.2

Note that if for the inequality $\alpha x \geq 2$ there exists a triplet $N_1 \subset J_1$ such that J_0 is N_1-maximal and (2.1) is a minimal valid inequality for $P_I(A^{N_1 \cup J_0})$, then in the 2-cover graph of $A^{J}_{M(J_0)}$ every vertex of $J_1 \setminus N_1$ is connected to some vertex of N_1, and $\alpha x \geq 2$ can be obtained by the simplified lifting procedure of Corollary 3.7.

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